SELF-ADJOINT OPERATORS AND NONTRIVIAL ZEROS OF DIRICHLET L-FUNCTION

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ABSTRACT. We give a kind of self-adjoint operator, whose spectrums are the set $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho$ is nontrivial zeros of L-function $L(\chi, s)\}$.

Alain Connes[1] first proved the critical zeros of Hecke L-function corresponds to the the spectrums of a suitable operator. His main methods is to deal with the relation between the L^2 of adele class space and the L^2 of idele class space by the adelic Riemann-Roth theorem. In [4, Thm.4.16], Connes, Consani and Marcolli give the spectral realization of zeros of Dirichlet L-function as the action of $\mathbb{R}^*_+$ on a suitable space.

Motivated by Alain Connes's spectral interpretation for the zeros of L-functions, Ralf Meyer[8] developed an alternative spectral interpretation for the poles and zeros and for André Weil's explicit formula, but no longer directly related to the Riemann hypothesis. André Weil's explicit formula also has relation with Riesz potential(see [6]). In [9, Corollary4.2], R. Meyer proved that the eigenvalues of the transpose D^t_- of the operator D_-(induced by D on some function space) acting on some space of continuous linear functionals are exactly the nontrivial zeros of $\zeta(s)$. Furthermore, Xian-Jin Li [7] proved that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-. His method has been generalized to Dirichlet L function by Dongsheng Wu[13]. Liming Ge, Xian-Jin Li, Dongsheng Wu and Boqing Xue in [5] proved that the correspondence between the set of eigenvalues of D_- acting on $\mathcal{H}$ and the set of nontrivial zeros of $\zeta(s)$ is one-to-one.

Inspired by the above results, we find a suitable self-adjoint operator which are related with the nontrivil zeros of Dirichlet L-function. The method we do here is an old idea, called Hilbert-Pólya conjecture.

1. THE SELF-ADJOINT OPERATOR

Denote $\mathbb{R}^\times_+ = (0, \infty)$, $C^\infty(\mathbb{R}^\times_+)$ the set of smooth complex valued functions on $\mathbb{R}^\times_+$. Let

$$\mathcal{H}_0 = \{f \in C^\infty(\mathbb{R}^\times_+) \mid \lim_{x \to \infty} x^m f^{(n)}(x) = 0 \text{ and } f^{(n)}(0) := \lim_{x \to 0^+} f^{(n)} \text{ exists}, \forall m, n \in \mathbb{N}\}.$$

$$\mathcal{H}_\cap := \{f \in \mathcal{H}_0 \mid \int_0^\infty f(x)dx = 0, f(0) = 0 \text{ and } f^{(2n+1)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

$$\mathcal{H}_- := \{f \in \mathcal{H}_0 \mid f^{(n)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

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Let χ be a primitive Dirichlet character. Define

$$\mathcal{H}_\cap^\chi := \{f \in \mathcal{H}_0 \mid f^{(2n+1)}(0) = 0 \text{ if } \chi(-1) = 1, f^{(2n)}(0) = 0 \text{ if } \chi(-1) = -1, \forall n \in \mathbb{N}\}.$$

The inner product on $\mathcal{H}_0$ is defined by

$$\langle f(x), g(x) \rangle = \int_0^\infty f(x)\overline{g(x)}dx.$$

Then $\mathcal{H}_0$ is a unitary space, i.e., a complex space with inner product. We define two self-adjoint operators $\mathcal{D}, \mathcal{M}$ on $\mathcal{H}_0$ by

$$\mathcal{D}f(x) = -if'(x), \quad \mathcal{M}f(x) = xf(x).$$

That is, for $f, g \in \mathcal{H}_0$ we have

$$\langle \mathcal{D}f, g \rangle = \langle f, \mathcal{D}g \rangle, \quad \langle \mathcal{M}f, g \rangle = \langle f, \mathcal{M}g \rangle.$$

The above equations are easy to compute, leaving it to the reader, or see [**?**, Example7.1.5,7.1.6]. It is easy to check that

(1.1) $$\mathcal{M}\mathcal{D} - \mathcal{D}\mathcal{M} = i.$$

Our key idea is the following

Theorem 1.1. $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on $\mathcal{H}_0$.

Proof. From equation(1.1), for $f, g \in \mathcal{H}_0$ we have

$$\langle \mathcal{M}\mathcal{D}f, g \rangle = \langle f, \mathcal{M}\mathcal{D}g \rangle + i\langle f, g \rangle.$$

Then

$$\begin{aligned}
\langle (\mathcal{M}\mathcal{D} - \frac{i}{2})f, g \rangle &= \langle \mathcal{M}\mathcal{D}f, g \rangle - \frac{i}{2}\langle f, g \rangle \\
&= \langle f, \mathcal{M}\mathcal{D}g \rangle + i\langle f, g \rangle - \frac{i}{2}\langle f, g \rangle \\
&= \langle f, \mathcal{M}\mathcal{D}g \rangle + \frac{i}{2}\langle f, g \rangle \\
&= \langle f, (\mathcal{M}\mathcal{D} - \frac{i}{2})g \rangle.
\end{aligned}$$

Hence, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator. $\qquad\square$

Lemma 1.2. $\mathcal{H}_-$ is invariant subspace of $\mathcal{D}, \mathcal{M}$, hence, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on it.

Proof. It is easy to check that for $f \in \mathcal{H}_-$, we have $\mathcal{D}f, \mathcal{M}f \in \mathcal{H}_-$. Hence, $\mathcal{D}, \mathcal{M}$ are operators on $\mathcal{H}_-$. Also, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on it. $\qquad\square$

For $f \in \mathcal{H}_\cap$, define the operator $\mathcal{Z}$ by

$$(\mathcal{Z}f)(x) = \sum_{n=1}^\infty f(nx),$$

and for $f \in \mathcal{H}_\cap^\chi$, define the operator $\mathcal{Z}_\chi$ by

$$(\mathcal{Z}_\chi f)(x) = \sum_{n=1}^\infty \chi(n)f(nx).$$

Then we have $\mathcal{Z}\mathcal{H}_\cap \subset \mathcal{H}_-, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi \subset \mathcal{H}_-$ (see [13, Thm.2.9]). Let $(\mathcal{Z}\mathcal{H}_\cap)^\perp, (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp$ be the orthogonal complement in $\mathcal{H}_-$, i.e., the set of vectors which is orthogonal to $\mathcal{Z}\mathcal{H}_\cap$ (resp. $\mathcal{Z}_\chi \mathcal{H}_\cap^\chi$). Then we have the following conjecture

Conjecture 1.3.

$$(1.2) \qquad \mathcal{H}_- = (\mathcal{Z}\mathcal{H}_\cap)^\perp \bigoplus \mathcal{Z}\mathcal{H}_\cap = (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp \bigoplus \mathcal{Z}_\chi \mathcal{H}_\cap^\chi.$$

Theorem 1.4. *Under Conjecture*(1.3)*, we have the canonical isomorphisms*

$$(\mathcal{Z}\mathcal{H}_\cap)^\perp \simeq \mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap, \quad (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp \simeq \mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi.$$

Moreover, $\mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap$ and $\mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are unitary spaces. Further, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of them.

Proof. Since $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are invariant spaces of $\mathcal{M}\mathcal{D} - \frac{i}{2}$ and $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of $\mathcal{H}_-$, we have $(\mathcal{Z}\mathcal{H}_\cap)^\perp, (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp$ are invariant spaces of $\mathcal{M}\mathcal{D} - \frac{i}{2}$. Furthermore, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of them. □

Remark 1.5. $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are invariant spaces of $\mathcal{M}\mathcal{D}$, but they are not invariant spaces under $\mathcal{M}$ or $\mathcal{D}$ lonely.

Theorem 1.6. *The Riemann hypothesis is true under Conjecture*(1.3)*.*

Proof. Let $S = \{i(\rho - \frac{1}{2}) \mid \rho \text{ is nontrivial zeros of } \zeta(s)\}$ and $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho \text{ is nontrivial zeros of } L(\chi, s)\}$. Then by Theorem1.2, Theorem1.3 in [13], the spectrum of $\mathcal{M}\mathcal{D} - \frac{i}{2}$ on $\mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap$ is S and on $\mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ is S_χ. Since $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator, we have $S, S_\chi \subset \mathbb{R}$, which implies the Riemann hypothesis. □

2. Connes's method

Let $\mathbb{R}_+^*$ be the mulplicative group of positive real numbers and $L^2(\mathbb{R}_+^*)$ be the Hilbert space of square integral complex valued functions of $\mathbb{R}_+^*$ with respect to the Haar measure d^*x on $\mathbb{R}_+^*$. We consider the smooth function space with compact support $C_c^\infty(\mathbb{R}_+^*)$. Let $C_c^\infty(\mathbb{R}_+^*)_0$ be the subspace of $C_c^\infty(\mathbb{R}_+^*)$ consisting of those $f \in C_c^\infty(\mathbb{R}_+^*)$ such that $\int_{\mathbb{R}_+^*} x f(x) d^*x = 0$.

Let $L^2(\mathbb{R})_{\mathrm{ev}}$ be the Hilbert space of square integrable even functions on $\mathbb{R}$. The inner product in $L^2(\mathbb{R})_{\mathrm{ev}}$ is normalized as follows

$$\langle \eta, \xi \rangle := \frac{1}{2} \int_\mathbb{R} \eta(x) \overline{\xi(x)} dx = \int_0^\infty \eta(x) \overline{\xi(x)} dx.$$

Then $L^2(\mathbb{R})_{\mathrm{ev}}$ is isomorphic to $L^2(\mathbb{R}_+^*)$ by the unitary isomorphism(see [3, Equation17])

$$w : L^2(\mathbb{R})_{\mathrm{ev}} \to L^2(\mathbb{R}_+^*), \quad (w\xi)(\lambda) := \lambda^{\frac{1}{2}} \xi(\lambda).$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$. Denote by $\mathcal{S}(\mathbb{R}_+^*) := w(\mathcal{S}(\mathbb{R}) \cap L^2(\mathbb{R})_{\mathrm{ev}})$ the Schwartz space of $\mathbb{R}_+^*$.

Let $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to S^1$ be a Dirichlet character. Similar to the map $\mathfrak{C}$ in [2, Chp.5,§6], we define the following map(also appearing in [9, §6])

$$\Sigma_\chi : C_c^\infty(\mathbb{R}_+^*)_0 \to L^2(\mathbb{R}_+^*), \quad f(x) \mapsto \sum_{n=1}^\infty \chi(n) f(nx).$$

Then $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ is a subspace of $L^2(\mathbb{R}_+^*)$.

Consider the regular representation U of $\mathbb{R}_+^*$ on $L^2(\mathbb{R}_+^*)$:

$$(U(\lambda)f)(x) := f(\lambda^{-1}x), \quad f \in L^2(\mathbb{R}_+^*), \ \lambda \in \mathbb{R}_+^*.$$

Let $U(\lambda)^t$ be the transpose of $U(\lambda)$, i.e., $(U(\lambda)^t f)(x) = f(\lambda x)$. Then one has $\langle U(\lambda)f, g \rangle = \langle f, U(\lambda)^t g \rangle$, which is obtained by

$$\int_{\mathbb{R}_+^*} f(\lambda^{-1}x)\overline{g(x)}d^*x = \int_{\mathbb{R}_+^*} f(y)\overline{g(\lambda y)}d^*y.$$

Since $\mathbb{R}_+^*$ is one-parameter group on $L^2(\mathbb{R}_+^*)$, the representation U is generated by a unique unbounded operator $D($ see [12, Thm.6.2]$)$ with

$$Df(x) = \lim_{h \to 0} \frac{f(e^{-h}x) - f(x)}{h} = -x\frac{\partial}{\partial x}f(x),$$

which is similar to the case appearing in [1, III,equation(26)]. By Theorem1.1, we know $i(D - \frac{1}{2})$ is a self-adjoint operator. The operator D acting on $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ is closed, hence D can act on $\mathcal{H}_\chi := (\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0)^\perp$, which is denoted the orthogonal of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ in $\mathcal{S}(\mathbb{R}_+^*)$. We denote restriction of D on the subspace $\mathcal{H}_\chi$ by D_χ. Since we deal with the orthogonal of an invariant subspace, we can assume

(2.1) $$U^t(h)\eta = \eta,$$

for some $h \in C^\infty(\mathbb{R}_+^*)$ such that $\hat{h}$ has compact support. Here, $U^t(h)$ is an operator on $L^2(\mathbb{R}_+^*)$ defined by

$$U^t(h)f(x) = \int_{\mathbb{R}_+^*} h(\gamma)f(\gamma x)d^*\gamma.$$

Then we have the following theorem, where the idea follows Connes' result[1, III.Thm1].

Theorem 2.1. *For the Hilbert space $\mathcal{H}_\chi$, the operator D_χ has discrete spectrum, denoted by $\mathrm{Sp}D_\chi$. In fact, $\mathrm{Sp}D_\chi$ is just the set of the notrivial zeros of L-function $L(\chi, s)$.*

Proof. Take $\psi \in \mathcal{S}(\mathbb{R}_+^*)$. Then ψ is in the orthogonal of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ if and only if

(2.2) $$\int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x)\overline{\psi(x)}d^*x = 0, \quad \forall f \in C_c^\infty(\mathbb{R}_+^*)_0.$$

Consider the pairing $\mathbb{R}_+^* \times \mathbb{R} \to S^1$, $(r,t) \mapsto r^{it}$. Under this pairing, $\mathbb{R}$ can be viewed as the character group of $\mathbb{R}_+^*$. Consider the Fourier transform $\hat{\psi}(t) = \int_0^\infty \overline{\psi(x)}x^{-it-1}dx = \int_{\mathbb{R}_+^*} \overline{\psi(x)}x^{-it}d^*x$ (see [3, Equation21], or [11, Definition in §3.3, P.102]), which is a meromorphic function over $\mathbb{C}$ and its only singularities are simple poles at a subset of non-positive integers [13, Lemma2.1]. Then the inverse Fourier transform(see [11, Theorem3.9]) is given by

$$\overline{\psi(x)} = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{\psi}(t)x^{it}dt.$$

Put the above formula into (2.2), we formally have

$$\int_{\mathbb{R}_+^*} \int_{-\infty}^{\infty} (\Sigma_\chi f)(x) x^{it} \hat{\psi}(t) d^*x dt = \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^*} \left(x^{it} \sum_{n=1}^{\infty} \chi(n) f(nx) \right) \hat{\psi}(t) d^*x dt$$

(2.3)
$$= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{it}} \hat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it} d^*y dt$$

$$= \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^*} L(\chi, it) \hat{\psi}(t) f(y) y^{it} d^*y dt$$

In the above formula,

$$\int_{\mathbb{R}_+^*} \left(x^{it} \sum_{n=1}^{\infty} \chi(n) f(nx) \right) \hat{\psi}(t) d^*x = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{it}} \hat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it} d^*y$$

holds when $\Im(t) < -1$. If χ is not trivial character, we have the analytic continuation for $t \in \mathbb{C}$

$$\int_{\mathbb{R}_+^*} \left(x^{it} \sum_{n=1}^{\infty} \chi(n) f(nx) \right) \hat{\psi}(t) d^*x = L(\chi, it) \hat{\psi}(t) \int_{\mathbb{R}_+^*} f(y) y^{it} d^*y$$

Then by Fubini theorem, we know (2.3) holds for nontrivial character χ, that is,

$$\int_{\mathbb{R}_+^*} \int_{-\infty}^{\infty} (\Sigma_\chi f)(x) x^{it} \hat{\psi}(t) d^*x dt = \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^*} L(\chi, it) \hat{\psi}(t) f(y) y^{it} d^*y dt.$$

If χ is trivial, then $L(\chi, it)$ has a pole at $t = -i$ and we should use the condition

(2.4)
$$\int_{\mathbb{R}_+^*} f(y) y^{i(-i)} d^*y = \int_0^{\infty} f(y) dy = 0.$$

But the space of function $f(x)x \in C_c^{\infty}(\mathbb{R}_+^*)$ such that (2.4) holds is still dense in the Schwartz space $\mathcal{S}(\mathbb{R}_+^*)$.

Combine (2.2) and (2.3), one has

(2.5)
$$L(\chi, it) \hat{\psi}(t) = 0$$

Since $L(\chi, it)$ is an analytic function of t, we see that it is a multiplier of the algebra $\mathcal{S}(\mathbb{R})$ of Schwartz functions in the variable t. Moreover, $|L(\chi, it)| = O(|t|^N)$ (see [10, 5.3]). Thus the product $L(\chi, it) \hat{\psi}(t)$ is still a tempered distribution, and so is its Fourier transform. If the latter vanishes when tested on arbitrary functions which are smooth with compact support, then $L(\chi, it) \hat{\psi}(t)$ vanishes.

To understand the equation (2.5), let consider an equation for distributions $\alpha(t)$ on S^1 of the form

(2.6)
$$\varphi(t) \alpha(t) = 0.$$

We assume $\varphi(t) \in C^{\infty}(S^1)$ has finitely many zeros x_i of finite order n_i. Let J be the ideal of $C^{\infty}(S^1)$ generated by φ. Then one has $\psi \in J \Leftrightarrow$ the order of ψ at x_i is no less than n_i [1, P.86]. Thus the distributions $\delta_{x_i}, \delta'_{x_i}, \ldots, \delta_{x_i}^{n_i-1}$ form a basis of the space solutions of (2.6).

Now $\hat{\psi}(t)$ is a distribution with compact support such that ψ is orthogonal to $\Sigma_\chi C_c^{\infty}(\mathbb{R}_+^*)_0$ and $L(\chi, it) \hat{\psi}(t) = 0$. Then we have $\hat{\psi}$ is a finite linear combination of the distributions

$$\delta_t^{(k)}, \quad L(\chi, it) = 0, \quad k < \text{order of the zero.}$$

Conversely, let s be a zero of $L(\chi, s)$ and $k > 0$ its order. Define

$$(2.7) \qquad \Delta_s(f) := \int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x) x^s d^* x.$$

For $f \in C_c^\infty(\mathbb{R}_+^*)_0$, we have

$$\Delta_s(f) = \sum_{n=1}^\infty \chi(n) \int_{\mathbb{R}_+^*} f(nx) x^s d^* x$$

$$= \sum_{n=1}^\infty \frac{\chi(n)}{n^s} \int_{\mathbb{R}_+^*} f(y) y^s d^* y$$

$$= L(\chi, s) \int_{\mathbb{R}_+^*} f(y) y^s d^* y.$$

Hence,

$$\left(\frac{d}{ds} \right)^a \Delta_s(f) = 0, \quad a = 0, 1, \ldots, k-1.$$

Differentiate the equation(2.7), one has

$$\left(\frac{d}{ds} \right)^a \Delta_s(f) = \int_{\mathbb{R}_+^*} (\Sigma_\chi f)(x) x^s (\log x)^a d^* x$$

Thus η belongs to the orthogonal of $\Sigma_\chi C_c^\infty(\mathbb{R}_+^*)_0$ and satisfies (2.1) iff it is a finite linear combination of functions of the form

$$\eta_{t,a}(x) = x^{it} (\log x)^a,$$

where

$$L(\chi, it) = 0, \quad a < \text{order of the zero}.$$

The transpose of the representation U is thus given in the above basis by

$$U(\lambda)^t \eta_{t,a}(x) = \eta_{t,a}(\lambda x)$$

$$= (\lambda x)^{it} (\log(\lambda x))^a$$

$$= (\lambda x)^{it} (\log(\lambda) + \log(x))^a$$

$$(2.8) \qquad = \sum_{b=0}^a C_a^b \lambda^{it} (\log \lambda)^b \eta_{t,a-b}(x).$$

The multiplication operator by a function with bounded derivatives is a bounded operator in any Sobolev space. Using the density in the orthogonal of the range of Σ_χ of vectors satisfying (2.1), one can check that if $L(\chi, is) \neq 0$, then is does not belong to the spectrum of D_χ^t. This determines the spectrum of the operator D_χ^t and hence of its transpose D_χ. $\qquad \square$

Corollary 2.2. *If Theorem2.1 is true, then the Riemann hypothesis is true.*

Proof. Since $i(D_\chi - \frac{1}{2})$ is self-adjoint operator on the inner space $\mathcal{H}_\chi$, the theorem follows by Theorem2.1. $\qquad \square$

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